Computational Method 2 Spring 2025

HW3

Q1 Dual of 3D Ising model

a) Show that the low temperature expansion of the 3D Ising model is equivalent to the high temperature expansion of a 3D Z(2) gauge theory. (work out the first 2 correction terms)

b) What about the high temperature expansion of the 3D Ising model? Is it equivalent to the low temperature expansion of the Z(2) gauge theory?

c) Argue that the the dual of 4D Z(2) gauge theory is another Z(2) gauge theory.

Q2 Heat Capacity

a) Show that the heat capacity

$$c = \frac{\partial}{\partial T}\epsilon$$

where $\epsilon = \langle E\rangle/N$, can be obtained from measuring the fluctuation of the total energy, i.e.

$$c = \beta^2 \frac{1}{N}\left(\langle E^2\rangle - \langle E\rangle^2\right).$$

b) Write a numerical code for solving Ising Model. Numerically realize the equivalence of the above relations for Ising Model in 4D.

Q3 Numerical integration with random distribution

a) Compute the value of π in the following manner:

Perform N trials of this task: select two random numbers in the range of 0:1, accept the pairs if the sum of their square is smaller than 1.0. Finally, evaluate

$$A = \frac{N_{\text{accepted}}}{N_{\text{trials}}}.$$

What is the expected result?

b) Generate a non-uniform distribution of random number x ($N > 35000$) in the range of 0.0 : 4.0 according to the distribution

$$\rho(x) = e^x$$

Plot a normalized histogram H of your collection of x, and verify that it follows the distribution

$$H = \rho(x)/\mathrm{norm}$$

where

$$\mathrm{norm} = \int_0^4 dx \rho(x).$$

c) Use the random collection to compute an integral, i.e.,

$$\frac{1}{N} \sum \tilde{f}(x) \approx \frac{\int dx\, \tilde{f}(x)\, \rho(x)}{\int dx\, \rho(x)}.$$

Choose, e.g. $\tilde{f}(x) = e^{-2x}$ and verify that

$$\frac{\mathrm{norm}}{N} \sum \tilde{f}(x) \approx \int_0^4 dx e^{-x}.$$

d) What do we get when we choose $\tilde{f} = \rho^{-1}$? Compute the result.

Q4 Average distance in a box.

If we randomly pick 2 points in an N-dim cube, what would be the average distance? (Take box size to be 1.0 for simplicity.)

a) For 1D, the analytic solution is given by

$$\frac{\int dx_1 dx_2 |x_1 - x_2|}{\int dx_1 \int dx_2}.$$

Compute the integral. (ans. 1/3) (OMG!)

Verify this result by random sampling.

b) Compute the results for N = 2 to 10. (List them in a table.)

Q5 Moments, Cumulants and all that

$W = \ln Z$ is extensive.

Introducing an external field h to the integral

$$Z(h) = \int_{-\infty}^{\infty} dx\, e^{-V(x^2+x^4)+V\,h\,x},$$

and define

$$Z_n = \frac{1}{Z(h)}\,\partial_h^n Z(h) = \langle x^n \rangle \times V^n$$

and

$$W(h) = \ln Z(h)$$
$$W_n = \partial_h^n W(h).$$

a) Show that

$$\begin{aligned}
W_1 &= Z_1 \\
W_2 &= Z_2 - Z_1^2 \\
W_3 &= Z_3 - 3Z_1 Z_2 + 2Z_1^3 \\
W_4 &= Z_4 - 4Z_1 Z_3 - 3Z_2^2 + 12 Z_1^2 Z_2 - 6Z_1^4
\end{aligned}$$

b) Verify these relations numerically, i.e. taking numerical derivatives on $W(h)$ and performing corresponding numerical integrations

$$Z_n = \frac{1}{Z(h)} \int_{-\infty}^{\infty} dx\, V^n x^n\, e^{-V(x^2+x^4)+V\,h\,x}.$$

Plot them as functions of h in the range of -2:2. (Take V = 2.)

c) Verify the linked cluster theorem: $W_n \propto V$ at large V, i.e. $W_n/V = w_n$ becomes intensive.