Computational Method 2 Spring 2025

HW1

Q1 Residue theorem.

The Cauchy's residue theorem states that the line integral of an analytic function over a closed path Γ is given by ($2\pi i$ times) the sum of residues:

$$\oint_\Gamma dz f(x) = 2\pi i \sum_{\text{Res.}} f(z_n).$$

- Take

  $$f(z) = \frac{1}{z^2+1}$$

  and verify the residue theorem by numerically computing the integral along the paths:

  - A) a square centered around the pole $z = i$;
  - B) an upper semi-circle of radius $R = 2$. Separate the contributions from the real line: -R:R and the arc.

- Repeat the calculation in B) for a larger R. Show that the arc contribution decreases as $R \to \infty$ and only the real-line contribution remains, hence

  $$\int_{-\infty}^{\infty} dx \, \frac{1}{x^2+1} = \pi.$$

Q2 Schwinger proper time regularization.

Consider the following identities:

$$\begin{aligned}
\mathcal{A}^{-1} &= \int_0^\infty dt \, e^{-t\mathcal{A}} \\
\ln \mathcal{A} &= -\int_0^\infty dt \, \frac{1}{t} \left(e^{-t\mathcal{A}} - e^{-t\mathcal{I}} \right).
\end{aligned}$$

- With $\mathcal{A} \to p_4^2 + \omega^2$, show (again!) that

  $$\int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \frac{1}{p_4^2+\omega^2} = \frac{1}{2\sqrt{\omega^2}}.$$

- The second identity is useful for regulating divergent integral. Consider

$$
\begin{aligned}
W[\omega;\Lambda] &= \int_{-\infty}^{\infty} \frac{dp_4}{2\pi}\, \ln\left(p_4^2+\omega^2\right) \\
&\to \int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \int_{1/\Lambda^2}^{\infty} dt\,(-\frac{1}{t})\, e^{-t(p_4^2+\omega^2)} + C \\
&= -\int_{1/\Lambda^2}^{\infty} dt\, \frac{1}{t^{\frac{3}{2}}}\, \frac{1}{2\sqrt{\pi}}\, e^{-t\omega^2} + C.
\end{aligned}
$$

Compute $W[\omega;\Lambda]$ numerically for a large enough Λ. You can forget about the integration constant C. Compare with the analytic result:

$$
W[\omega;\Lambda] = -\frac{\Lambda}{\sqrt{\pi}} + \omega + \mathcal{O}(1/\Lambda).
$$

- This suggests the definition of a physical W function:

$$
W_{\text{phys.}}[\omega] = \lim_{\Lambda\to\infty} \left(W[\omega;\Lambda] - W[0;\Lambda]\right) = \omega.
$$

  Re-derive the previous result via

$$
\int_{-\infty}^{\infty} dx\, \frac{1}{x^2+\omega^2} = \frac{1}{2\omega}\frac{\partial}{\partial\omega} W_{\text{phys.}}[\omega].
$$

- Generate a 3 x 3 positive definite matrix $\mathcal{A}$. Compute the inverse and determinant using the above identities.

Q3 Logistic Map.

Consider the logistic map:

$$
x_{n+1} = r x_n \left(1 - x_n\right).
$$

- Produce the bifurcation diagram: for a given r within $(0,4)$, iterate (starting from any initial value within $(0,1)$), collect the end point(s) and plot them as a function of r.
- Derive an analytic result for the stable end points within $r < 3$. Plot it on top of the bifurcation diagram.
- For $r = 3.6$, collect all the points $\{x_j\}$ from iterations and plot them in a normalized histogram. This is called the invariant density.
- Plot the invariant density at $r \to 4$. Compare with the analytic result:

$$
\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}.
$$

- Derive the above analytic result for $\rho(x)$.